

Classical Zero-Point Radiation and Relativity: The Problem of Blackbody Radiation Revisited

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Abstract

The physicists of the early 20th century were unaware of two ideas which are vital to understanding some aspects of modern physics within classical theory. The two ideas are: 1) the presence of classical electromagnetic zero-point radiation, and 2) the importance of special relativity. In classes of modern physics today, the problem of blackbody radiation within classical physics is still described in the historical context of the early 20th century. However, the inclusion of classical zero-point radiation and of relativity now allows a completely satisfactory classical understanding of blackbody radiation with the Planck spectrum, as well as of some other aspects of modern physics. Here we sketch the current classical understanding of blackbody radiation, pointing out that thermodynamics allows the presence of classical zero-point radiation, and that use of *nonrelativistic* physics leads to the Rayleigh-Jeans spectrum while *relativistic* physics gives the Planck spectrum. The current textbooks of modern physics are a century out of date in presenting the connections between classical and quantum physics.

I. INTRODUCTION

Current textbooks of modern physics are a century out-of-date in their discussions of the relations between classical and quantum physics. The textbooks still treat the connections in the historical context of the early 20th century.[1] Today it is well (but not widely) known that the classical physicists of the early 20th century were unaware of two crucial ideas vital to classical physics: 1) the presence of classical electromagnetic zero-point radiation and 2) the importance of relativity. When these two aspects are included, classical physics accounts for phenomena which the current texts regard as exclusively quantum phenomena. These phenomena include blackbody radiation, Casimir forces, van der Waals forces, harmonic oscillator behavior, decrease of specific heats at low temperatures, and diamagnetism.[2][3][4] In the present article, we review how the two crucial missing aspects transform the classical understanding of the blackbody radiation spectrum.

II. REVIEW OF THERMAL RADIATION IN CLASSICAL PHYSICS

1. Radiation normal Modes

Within classical physics, thermal radiation is treated as random classical electromagnetic cavity radiation which is invariant under scattering. Choosing for simplicity a rectangular, conducting-walled cavity of dimensions a , b , d , the radiation inside can be written as a sum over the radiation normal modes with vanishing scalar potential Φ and with vector potential \mathbf{A} given by[5]

$$\begin{aligned} \mathbf{A}(x, y, z, t) = \sum_{l,m,n=0}^{\infty} \sum_{\lambda=1}^2 q_{lmn,\lambda} c \left(\frac{32\pi}{abd} \right)^{1/2} \left\{ \hat{i} \varepsilon_{lmnx}^{(\lambda)} \cos \left(\frac{l\pi x}{a} \right) \sin \left(\frac{m\pi y}{b} \right) \sin \left(\frac{n\pi z}{d} \right) \right. \\ + \hat{j} \varepsilon_{lmny}^{(\lambda)} \sin \left(\frac{l\pi x}{a} \right) \cos \left(\frac{m\pi y}{b} \right) \sin \left(\frac{n\pi z}{d} \right) \\ \left. + \hat{k} \varepsilon_{lmnz}^{(\lambda)} \sin \left(\frac{l\pi x}{a} \right) \sin \left(\frac{m\pi y}{b} \right) \cos \left(\frac{n\pi z}{d} \right) \right\} \end{aligned} \quad (1)$$

where $\hat{\varepsilon}_{lmn}^{(\lambda)}$ with $\lambda = 1, 2$ are the mutually orthogonal unit vectors satisfying $\varepsilon_x l + \varepsilon_y m + \varepsilon_z n = 0$, where $q_{lmn,\lambda}$ is the time-varying amplitude of the mode, and where the frequency of the mode is given by $\omega_{lmn} = c\pi(l^2/a^2 + m^2/b^2 + n^2/d^2)^{1/2}$. The radiation energy in the box is given by $\mathcal{E} = [1/(8\pi)] \int \int \int dx dy dz (E^2 + B^2)$ where $\mathbf{E} = -\nabla\Phi - (1/c)\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$,

so that

$$\mathcal{E} = \sum_{l,m,n=0}^{\infty} \sum_{\lambda=1}^2 (1/2) (\dot{q}_{lmn,\lambda}^2 + \omega_{lmn,\lambda}^2 q_{lmn,\lambda}^2) = \sum_{l,m,n=0}^{\infty} \sum_{\lambda=1}^2 J_{lmn,\lambda} \omega_{lmn,\lambda} \quad (2)$$

where $J_{lmn,\lambda}$ is the action variable $J = \int p dq$ of the mode,[6] $J_{lmn,\lambda} = \mathcal{E}_{lmn,\lambda} / \omega_{lmn,\lambda}$, and $\mathcal{E}_{lmn,\lambda}$ is the energy of the mode. Thus the energy of thermal radiation in a cavity can be expressed as a sum over the energies of the normal modes of oscillation, with each mode taking the form of a harmonic oscillator[7]

$$\mathcal{E} = (1/2) (\dot{q}^2 + \omega^2 q^2) \quad (3)$$

2. Thermodynamics of the Simple Harmonic Oscillator

Now the thermodynamics of a harmonic oscillator takes a particularly simple form because the system has only two thermodynamic variables T and ω . [8] In thermal equilibrium with a bath, the average oscillator energy is denoted by $U = \langle \mathcal{E} \rangle = \langle J \rangle \omega$, and satisfies $dQ = dU - dW$ with the entropy S satisfying $dS = dQ/T$. Now since J is an adiabatic invariant,[6] the work done on the system is given by $dW = \langle J \rangle d\omega = (U/\omega) d\omega$. Combining these equations, we have $dS = dQ/T = [dU - (U/\omega) d\omega]/T$. Writing the differentials in terms of T and ω , we have $dS = (\partial S/\partial T) dT + (\partial S/\partial \omega) d\omega$ and $dU = (\partial U/\partial T) dT + (\partial U/\partial \omega) d\omega$. Therefore $\partial S/\partial T = (\partial U/\partial T)/T$ and $\partial S/\partial \omega = [(U/\omega) + (\partial U/\partial \omega)]/T$. Now equating the mixed second partial derivatives $\partial^2 S/\partial T \partial \omega = \partial^2 S/\partial \omega \partial T$, we have $(\partial^2 U/\partial \omega \partial T)/T = (\partial U/\partial T)/\omega + (\partial^2 U/\partial T \partial \omega)/T - [(U/\omega) + (\partial U/\partial \omega)]/T^2$ or $0 = (\partial U/\partial T)/\omega - [(U/\omega) + (\partial U/\partial \omega)]/T^2$. The general solution of this equation is

$$U = f(T/\omega) \omega = \langle J \rangle \omega \quad (4)$$

where $f(T/\omega)$ is an unknown function which corresponds to the average value $\langle J \rangle$ of the action variable of the mode. When applied to thermal radiation, the result obtained here purely from thermodynamics corresponds to the familiar Wien displacement law of classical physics.

3. Possibility of Zero-Point Radiation

The energy expression (4) for an electromagnetic radiation mode (or for a harmonic oscillator) in thermal equilibrium allows two limits which make the energy independent from one of its two thermodynamics variables. When the temperature T becomes very large, $T \gg \omega$, so that the argument of the function $f(T/\omega)$ is large, the average energy U of the mode becomes independent of ω provided $f(T/\omega) \rightarrow \text{const}_1 \times T/\omega$ so that

$$U = f(T/\omega)\omega \rightarrow \text{const}_1 \times (T/\omega) \times \omega = \text{const}_1 \times T \quad \text{for } T/\omega \gg 1. \quad (5)$$

This is the familiar high-temperature limit where we expect to recover the Rayleigh-Jeans equipartition limit. Therefore we choose this constant as $\text{const}_1 = k_B$ corresponding to Boltzmann's constant. With this choice, our thermal radiation now goes over to the Rayleigh-Jeans limit for high temperature or low frequency.

In the other limit of small temperature, $T \ll \omega$, the dependence on temperature is eliminated provided $f(T/\omega) \rightarrow \text{const}_2$, so that

$$U = f(T/\omega)\omega \rightarrow \text{const}_2 \times \omega \quad \text{for } T/\omega \ll 1. \quad (6)$$

At this point, any theoretical description of thermal radiation must make a choice. If we choose this second constant to vanish, $\text{const}_2 = 0$, then this limit does not force us to introduce any constant beyond Boltzmann's constant which entered for the high-temperature limit of thermal radiation. On the other hand, if we choose a non-zero value for this constant, $\text{const}_2 \neq 0$, then we are introducing a second constant into the theory of thermal radiation, which constant has different dimensions from those of Boltzmann's constant. The units of this new constant const_2 correspond to *energy times length*. Furthermore, the choice of a non-zero value for this constant means that at temperature $T = 0$, there is random, temperature-independent radiation present in the system. This random radiation which exists at temperature $T = 0$ is classical electromagnetic zero-point radiation.

We emphasize that thermodynamics allows classical zero-point radiation within classical physics. The physicists of the early 20th century were not familiar with the idea of classical zero-point radiation, and so they made the choice $\text{const}_2 = 0$ which excluded the possibility of classical zero-point radiation. In his monograph on classical electron theory, Lorentz makes the explicit assumption that there is no radiation present at $T = 0$. [9] Today, we

know that the exclusion of classical zero-point radiation is a poor choice. However, the current textbooks of modern physics continue to present only the outdated, century-old classical view.

Once the possibility of classical zero-point radiation is introduced into classical theory, one looks for other phenomena where the zero-point radiation will play a crucial role. In particular, the (Casimir) force between two uncharged conducting parallel plates will be influenced by the presence of classical electromagnetic zero-point radiation.[2] By comparing theoretical calculations with experiments, one finds that the scale constant for classical zero-point radiation appearing in Eq. (6) must take the value $const_2 = 1.05 \times 10^{-34}$ Joule-sec. However, this value corresponds to the value of a familiar constant in physics; it corresponds to the value $\hbar/2$ where \hbar is Planck's constant. Thus in order to account for the experimentally observed Casimir forces between parallel plates, the scale of classical zero-point radiation must be such that $const_2 = \hbar/2$, and for each normal mode, the average energy becomes

$$U = f(T/\omega) \rightarrow (\hbar/2)\omega \quad \text{for } T \rightarrow 0. \quad (7)$$

We emphasize that Planck's constant enters classical electromagnetic theory as the scale factor in classical electromagnetic zero-point radiation. There is no connection whatsoever to any idea of quanta. Many students are misled by the textbooks of modern physics and regard Planck's constant as a "quantum constant." [10] This is a completely misleading idea. A physical constant is a numerical value associated with certain aspects of nature; the constant may appear in several different theories, just as Cavendish's constant G appears in both Newtonian physics and also in general relativity. Here we emphasize that Planck's constant \hbar appears in both classical and quantum theories.

III. ROLE OF RELATIVITY

4. *Nonrelativistic Classical Physics Gives the Rayleigh-Jeans Spectrum*

Today textbooks of modern physics present the Rayleigh-Jeans spectrum as though it were the unique result of classical physics.[1] Specifically, the equipartition-theorem result of *nonrelativistic* classical statistical mechanics is chosen as the average energy of each normal mode of a harmonic oscillator and therefore for each normal mode of classical thermal

radiation. This energy-equipartition choice corresponds to the Rayleigh-Jeans result

$$U_{RJ} = k_B T \quad (8)$$

for each normal mode of radiation. As we have seen above, this choice is not forced by thermodynamics. Rather it is the use *nonrelativistic* classical physics, either from *nonrelativistic* statistical mechanics or from *nonrelativistic* scatters, which leads consistently to the Rayleigh-Jeans result.

Indeed the importance of *nonrelativistic* physics is evident in the scattering calculations for random classical radiation. Clearly thermal radiation should be stable under scattering by charged mechanical systems. This stability is intrinsic to the idea of thermal equilibrium. A small harmonic oscillator (dipole oscillator) scatters radiation without changing the frequency of the incident radiation.[4] Thus a dipole oscillator will enforce the isotropic nature of the radiation spectrum but will not enforce any particular radiation spectrum for random classical radiation. However, a *nonlinear* dipole oscillator will indeed force an equilibrium spectrum of random radiation; the equilibrium spectrum it enforces is the Rayleigh-Jeans spectrum.[11] Indeed, the classical zero-point radiation spectrum $U = (\hbar/2)\omega$ is unstable under scattering by a *nonrelativistic* nonlinear oscillator; the nonrelativistic scatterer pushes the zero-point spectrum toward the Rayleigh-Jeans spectrum.

During the years around 1910, the question of the blackbody radiation spectrum was discussed by the leading physicists of the day. It was suggested that classical mechanics contained no constant comparable to Planck's constant \hbar appearing in the Planck blackbody radiation spectrum, and therefore presumably classical physics was incapable of producing the Planck spectrum.[12]

5. *Relativistic Classical Physics Gives the Planck Spectrum*

In Eq. (7) above, we pointed out that the thermodynamics of classical thermal radiation allows the possibility of classical electromagnetic zero-point radiation. Indeed, if one accepts classical zero-point radiation, then it turns out that the smoothest interpolation between the low-temperature zero-point radiation limit and the high-temperature equipartition limit is precisely the Planck spectrum including zero-point radiation[13] The energy per normal

mode for the Planck spectrum including zero-point radiation is given by

$$U_P = (\hbar\omega/2) \coth(\hbar\omega/(2k_B T)) = \hbar\omega/2 + \hbar\omega(\exp[\hbar\omega/k_B T] - 1)^{-1} \quad (9)$$

Clearly one may wonder how this smooth-interpolation result fits with the radiation scattering calculations.

It turns out that the spectrum of random classical electromagnetic zero-point radiation in Eq. (7) is invariant under adiabatic transformation. The spectrum is also Lorentz invariant, scale invariant, and conformal invariant.[14][15] Indeed the correlation functions of classical zero-point radiation involve no preferred length, time, velocity, or coordinate frame. The correlations involve only the geodesic separation between the spacetime points where correlations are evaluated.[16]

It is the Lorentz invariance of zero-point radiation which we wish to emphasize here. Today physicists believe that all physics is intrinsically relativistic; relativity is a meta theory to which all fundamental theories should conform. Thus the use of nonrelativistic scatterers is at odds with the Lorentz invariance of classical zero-point radiation and is also at odds with our expectation that physics should be relativistic.

Suppose that we insist that the charged mechanical scatterers of thermal radiation are *relativistic* scatterers. In this case, we encounter the relativistic conservation law[17] and also the no-interaction theorem of Currie, Jordan, and Sudershan.[18] The only relativistic interactions between particles involve either point collisions or else interaction through a complete relativistic field theory. This theorem reminds us why elementary treatments of relativity deal with particle point collisions and never with particle interactions through a potential. The only potential which has been extended to a complete relativistic field theory is the Coulomb potential which is extended to charged particle interactions within classical electrodynamics.

Thus all the scattering calculations by *nonrelativistic* dipole systems are suspect. The one simple *relativistic* scattering system corresponds essentially to a hydrogen atom. And for the Coulomb potential extended to relativistic electrodynamics, nature indeed includes a fundamental mechanical constant with the units of *energy* times *time*, namely e^2/c , which is common to all interacting charged particles. The relativistic hydrogen atom has all the properties which suggest the possibility of equilibrium with classical zero-point radiation and thermal radiation at the Planck spectrum.[19] Indeed, in Goldstein's mechanics text,[20]

we find the relativistic energy \mathcal{E} for a particle of mass m in a Coulomb potential Ze^2/r in terms of action-angle variables J_2 and J_3 takes the form[21]

$$\frac{\mathcal{E}}{mc^2} = \left(1 + \left[\frac{J_3 c}{Ze^2} - \frac{J_2 c}{Ze^2} + \left\{ \left(\frac{J_2 c}{Ze^2} \right)^2 - 1 \right\}^{1/2} \right]^{-2} \right)^{-1/2} \quad (10)$$

We notice immediately that the constant Ze^2/c is a crucial parameter for the system. If $J_2 < Ze^2/c$, then the energy expression in Eq. (10) is no longer valid because it involves the square-root of a negative quantity. This is a reminder that the constant e^2/c is a crucial parameter in *relativistic* classical physics and that the orbits of relativistic motion are quite different from the nonrelativistic elliptical orbits. Indeed, if $J_2 < Ze^2/c$, then (neglecting radiation emission) the relativistic mechanical trajectories plunge into the Coulomb center while conserving energy and angular momentum.[22] Relativistic systems can be quite different from nonrelativistic systems.

6. Relativistic Physics and Gravitation

The enormous difference between relativistic and nonrelativistic thermodynamic systems is immediately evident if we consider a thermodynamic system in gravity. Indeed, Boltzmann derived the Maxwell velocity distribution for nonrelativistic particles in thermal equilibrium by considering thermal equilibrium for particles under a gravitational field. In nonrelativistic physics, gravity couples only to the masses of particles; it does not couple to kinetic energy or potential energy. Accordingly, the pressure of a nonrelativistic system in thermal equilibrium under gravity reflects the changing particle density with height, while the temperature remains constant throughout the system. By considering pressure equilibrium and nonrelativistic particle motion in the vertical direction, Boltzmann was led to the Maxwell velocity distribution.[23]

The situation in relativistic physics is quite different. In relativistic physics, gravity couples to all energy, including both kinetic energy and energy in the electromagnetic fields. Thus if we consider flat spacetime but go to a Rindler frame undergoing time-independent proper acceleration, then neither the equivalence-principle gravitational field nor the temperature of a thermal system in equilibrium can be constant with changing distance from the event horizon. At spatial coordinates which are lower in the gravitational field, the temperature is higher so that $(g_{00})^{1/2}T = \text{const}$, where $g_{\mu\nu}$ corresponds to the metric tensor for the

spacetime.[24] If we consider classical electromagnetic zero-point radiation to be present, then the correlation in time at a given spatial coordinate reflects the gravitational field at the point. By carrying out a scaling transformation in time within a Rindler frame, one changes from thermal equilibrium at zero temperature over to thermal equilibrium at a finite non-zero temperature. Finally, by maintaining constant the local temperature as one moves ever further from the event horizon, one can move to a region where the spacetime metric becomes the Minkowski metric of an inertial frame while maintaining the finite temperature spectrum. The thermal spectrum which one finds is exactly the Planck spectrum.[25] Thus a relativistic treatment indeed gives the Planck spectrum for thermal radiation. One notes that the derivation requires relativistic behavior at every step of the analysis.

By analyzing a nonrelativistic thermal situation under gravity, Boltzmann derived the thermal distribution for particles. By analyzing the relativistic situation for radiation, including zero-point radiation, under gravity, one derives the Planck spectrum for thermal radiation. The difference between the Rayleigh-Jeans spectrum and the Planck spectrum does not represent a difference between classical and quantum physics. Rather the difference between the Rayleigh-Jeans spectrum and the Planck spectrum for thermal radiation within classical physics corresponds to the use of nonrelativistic versus relativistic physics.

IV. DISCUSSION

In this short sketch, we have outlined the analysis of the blackbody radiation spectrum from a modern classical point of view which includes both classical zero-point radiation and notes the importance of relativity. The inclusion of classical electromagnetic zero-point radiation will affect many aspects of the physics of small objects where the strong zero-point radiation fluctuations at high frequencies become important. Thus classical electromagnetic zero-point radiation has been used to give detailed calculations of Casimir forces, van der Waals forces, classical oscillator behavior, specific heats of solids, and diamagnetism.[2] These calculations give results in complete agreement with the calculations of quantum physics. The presence of classical zero-point radiation also transforms the old problem of atomic collapse where a planetary electron is claimed to spiral into the nucleus as the electron accelerates and radiates away its energy. In the presence of classical zero-point radiation, the spiraling electron will perform an orbital Brownian motion where it absorbs

radiation from the random zero-point radiation field as well as emitting radiation during acceleration. Numerical simulations show that the electron in a Coulomb orbit does not collapse into the nucleus when classical zero-point radiation is present.[26][27]

When one reads the most recent textbooks of modern physics with knowledge of the current state of classical electromagnetic theory, one can only be struck by the irony of these texts. Today modern physics texts invariably start with a discussion of relativity. However, the texts then drop the subject completely. The texts give no suggestion that the ideas of relativity may have significance for the discussions of blackbody radiation and atomic physics which follow. The texts contain many references to nonrelativistic mechanics but not one mention of the no-interaction theorem of relativistic mechanics.

There are many close and fascinating connections between classical and quantum physics. Relativistic classical electron theory is currently the best classical approximation to quantum physics. Discussions which present an outdated view of classical physics do a disservice to students who will become the scientists of the future.

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- [1] See the discussion of any textbook of modern physics. For example, R. Eisberg and R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles 2nd ed.* (Wiley, New York 1985) or K. S. Krane, *Modern Physics 2nd ed.* (Wiley, New York 1996) or J. R. Taylor, C. D. Zafiratos, and M. A. Dubson, *Modern Physics for Scientists and Engineers 2nd ed.* (Pearson, New York, 2003) or S. T. Thornton and A. Rex, *Modern Physics for Scientists and Engineers* (Brooks/Cole, Cengage Learning, Boston, MA 2013).
 - [2] A short review of work involving classical zero-point radiation is given by T. H. Boyer, “Any classical description of nature requires classical electromagnetic zero-point radiation” *Am. J. Phys.* **79**, 1163-1167 (2011).
 - [3] A review of the work on classical electromagnetic zero-point radiation up to 1996 is provided by L. de la Pena and A. M. Cetto, *The Quantum Dice - An Introduction to Stochastic Electrodynamics* (Kluwer Academic, Dordrecht 1996).
 - [4] See also the review by T. H. Boyer, “Random electrodynamics: The theory of classical electrodynamics with classical electromagnetic zero-point radiation,” *Phys. Rev. D* **11**, 790-808 (1975).

- [5] This cavity situation is a problem in D. J. Griffiths, *Introduction to Electrodynamics 4th ed.* (Pearson, New York 2013), problem 9.40 on p. 435. See, for example, A. Zangwill, *Modern Electrodynamics* (Cambridge U. Press, New York 2013), p. 703 and E. A. Power, *Introductory Quantum Electrodynamics* (American Elsevier, New York 1964), p. 13-15.
- [6] Action-angle variables are discussed by, for example, H. Goldstein, *Classical Mechanics 2nd ed.* (Addison-Wesley, Reading, MA 1981), Sections 10-5 through 10-7, pp. 457-484. The action variables are adiabatic invariants.
- [7] See Zangwill and Power in ref. 5.
- [8] T. H. Boyer, “Thermodynamics of the harmonic oscillator: Wien’s displacement law and the Planck spectrum,” *Am. J. Phys.* **71**, 866-870 (2003).
- [9] Traditional classical electron theory is described by H. A. Lorentz, *The Theory of Electrons* (Dover, New York 1952). This volume is a republication of the second edition of 1915 based on Lorentz’s Columbia University lectures of 1909. On page 20 and on page 240, note 6, Lorentz gives his explicit assumption on the boundary conditions for Maxwell’s equations; the assumption excludes the possibility of classical zero-point radiation.
- [10] See the essay by T. H. Boyer, “Is Planck’s Constant h a ‘Quantum’ Constant? An Alternative Classical Interpretation,” arXiv 1301.6043.
- [11] J. H. Van Vleck intended to publish this result in the mid 1920s but was diverted by the appearance of Schroedinger’s quantum mechanics. (Private communication to the author from J. H. Van Vleck.) T. H. Boyer, “Equilibrium of random classical electromagnetic radiation in the presence of a nonrelativistic nonlinear electric dipole oscillator,” *Phys. Rev. D* **13**, 2832-2845 (1976) and “Statistical equilibrium of nonrelativistic multiply periodic classical systems and random classical electromagnetic radiation,” *Phys. Rev. A* **18**, 1228-1237 (1978).
- [12] In ref. 9, Lorentz returns repeatedly to the idea that Planck’s blackbody spectrum requires that some aspect of nature must be the same for all mechanical systems. See, for example, Section 58 on p. 78. In Section 75 on page 96, Lorentz notes the constancy of the Wien-displacement-theorem quantity $T\lambda_{\text{max}}$. He remarks, “[If we can account for this constant], we may hope to find in what manner the value of this constant is determined by some numerical quantity that is the same for all ponderable bodies.”
- [13] See the calculation in Section VIII on p. 868 of ref. 8.
- [14] The Lorentz invariance appears in T. W. Marshall, “Statistical Electrodynamics,” *Proc. Camb.*

- Phil. Soc. **61**, 537-546 (1965).
- [15] T. H. Boyer, “Derivation of the Blackbody Radiation Spectrum without Quantum Assumptions,” Phys. Rev. **182**, 1374-11383 (1969) and “Conformal Symmetry of Classical Electromagnetic Zero-Point Radiation,” Found. Phys. **19**, 349-365 (1989).
 - [16] T. H. Boyer, “Contrasting Classical and Quantum Vacuum States in Non-inertial Frames,” Found. Phys. **43**, 923-947 (2013).
 - [17] T. H. Boyer, “Illustrating some implications of the conservation laws in relativistic mechanics,” Am. J. Phys. **77**, 562-569 (2009).
 - [18] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, “Relativistic Invariance and Hamiltonian theories of interacting particles,” Rev. Mod. Phys. **34**, 350-375 (1963). The no-interaction theorem is referred to in Goldstein’s mechanics text ref. 6, on pages 332, 334, and 362.
 - [19] T. H. Boyer, “Blackbody radiation and the scaling symmetry of relativistic classical electron theory with classical electromagnetic zero-point radiation,” Found. Phys. **40**, 1102-1116 (2010).
 - [20] See H. Goldstein in ref. 6, p. 498, problem 28.
 - [21] Here we have chosen action-angle variables differing by a factor of $1/(2\pi)$ from those of problem 28 of Goldstein’s text and have written the strength of the potential as $k = Ze^2$.
 - [22] T. H. Boyer, “Unfamiliar trajectories for a relativistic particle in a Kepler or Coulomb potential,” Am. J. Phys. **75**, 992-997 (2004).
 - [23] Boltzmann’s derivation is discussed, for example, by R. Resnick and D. Halliday, *Physics* (Wiley, New York, 1967), supplementary topic iv, p. 11; R. Becker and G. Leibfried, *Theory of Heat 2nd. ed.* (Springer, New York, 1967), p. 94.
 - [24] R. C. Tolman, *Thermodynamics and Cosmology* (Dover, New York, 1987), p. 318; R. C. Tolman and P. Ehrenfest, Phys. Rev. **36**, 1791 (1930).
 - [25] T. H. Boyer, “The blackbody radiation spectrum follows from zero-point radiation and the structure of relativistic spacetime in classical physics,” Found. Phys. **42**, 595-614 (2012). See also, the related discussions by T. H. Boyer, “Derivation of the Planck spectrum for relativistic classical scalar radiation from thermal equilibrium in an accelerating frame,” Phys. Rev. D **81**, 105024 (2010) and “Classical physics of thermal scalar radiation in two spacetime dimensions,” Am. J. Phys. **79**, 644-656 (2011).
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- electrodynamics,” *Physics Letters A* **317**, 14-20 (2003).
- [27] T. M. Nieuwenhuizen and M. T. P. Liska, “Simulation of the hydrogen ground state in Stochastic Electrodynamics,” *Physica Scripta T***165**, 014006 (2015), (arXiv: 1502.06856v2) and “Simulation of the hydrogen ground state in Stochastic Electrodynamics-2,” *Found. Phys.* **45**, 1190-1202 (2015), (arXiv: 1506.06787v1). See also, T. H. Boyer, “Classical Zero-Point Radiation and Relativity; the Problem of Atomic Collapse Revisited,” submitted for publication, arXiv 1511.02083.